GIRRAWEEN HIGH SCHOOL

TRIAL EXAMINATION

2009

MATHEMATICS

EXTENSION 2

Time allowed - Two hours
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All questions are worth equal marks.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1 , Question 2 , etc. Each piece of paper must show your name.
- You may ask for extra pieces of paper if you need them.

Question 1 *(15 Marks)* **Marks**

Evaluate:

(a) $\int_1^e \frac{e^x}{1+e^{2x}}.dx$ **3**

(b) $\int \frac{1}{3+\cos x}.dx$ **3**

(c) $\int x\sin 2x\,.dx$ **2**

(d) Express $\frac{22-5x}{(x+1)(x-2)^2}$ in the form $\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$.

Hence find $\int \frac{22-5x}{(x+1)(x-2)^2}.dx$ **4**

(e) Evaluate $\int \frac{1}{x^2\sqrt{4-x^2}}.dx$ **3**

Question 2 *(15 Marks)*

(a) If $z=1+3i$ and $w=2-i$ find in $x+iy$ form: **3**

(i) $\bar{z}w$

(ii) $\frac{z}{w}$

(b) If $z=1-i\sqrt{3}$ express z^5 in modulus/argument form. **3**

Hence express z^5 in $x+iy$ form.

(c) Sketch on an Argand diagram where the inequalities **3**

$|z-1|\le 2$ and $\frac{\pi}{4}< Arg(z-i)<\frac{\pi}{2}$ hold simultaneously.

(d) z is an arbitrary complex number with $0<Arg\,z<\frac{\pi}{2}$. **6**

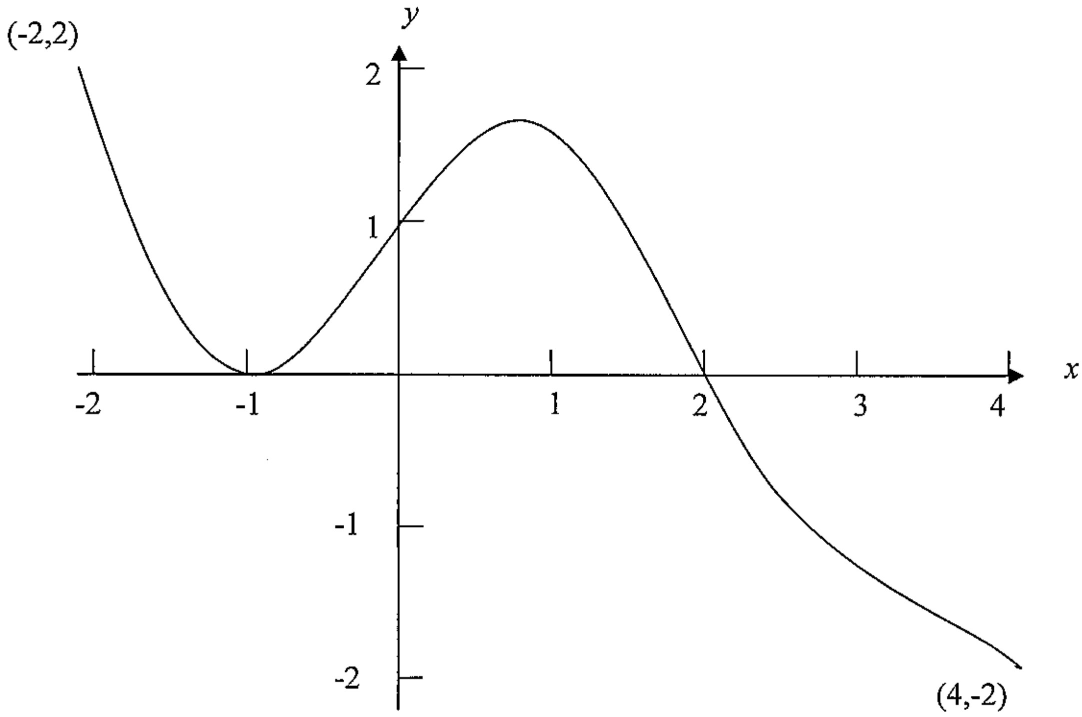
(i) Sketch z and iz on an Argand diagram.

(ii) Prove that $|iz-z|^2=2|z|^2$.

(iii) If A represents the point z, B represents the point iz and O represents the origin, find the centre and radius of the circle with A, B and O at the circumference if $z=6+8i$ (you may NOT use this value for z in parts (i) and (ii).

Question 3 *(15 Marks)* **Marks**

(a) Below is the graph of $y = f(x)$ from $x = -2$ to $x = 4$. **6**



Draw separate $\frac{1}{3}$ page diagrams of:

(i) $y = \dfrac{1}{f(x)}$

(ii) $y^2 = f(x)$

(iii) $y = e^{f(x)}$

(b) For the hyperbola $\dfrac{(x-1)^2}{9} - \dfrac{(y+2)^2}{16} = 1$ find: **9**

(i) The eccentricity.

(ii) The co-ordinates of the foci.

(iii) The equations of the directrices.

(iv) The equations of the asymptotes.

(v) Sketch the graph of the hyperbola $\dfrac{(x-1)^2}{9} - \dfrac{(y+2)^2}{16} = 1$

showing all of these features.

Question 4 *(15 marks)* **Marks**

(a) If the roots of the polynomial equation **8**

$P(x) = 3x^3 - 2x^2 + 5x + 1 = 0$

are α, β and δ:

(i) Find $\alpha^2 + \beta^2 + \delta^2$.

(ii) Find $\alpha^3 + \beta^3 + \delta^3$.

(iii) Explain why the equation $P(x) = 0$ has two complex (i.e. non real) roots.

(iv) Form the polynomial equation with roots α^2, β^2 and δ^2.

(b) If $I_n = \int \cos^n x.dx$, n a positive integer show that **4**

$$I_n = \frac{\cos^{n-1} x.\sin x}{n} + \frac{n-1}{n} I_{n-2}.$$

Hence find $\int_0^{\frac{\pi}{2}} \cos^{10} x.dx$

(c) Use the method of cylindrical shells to find the volume of the solid of revolution formed when the area enclosed by the curve $y = 10x - x^2 - 16$ and the x axis is rotated about the line $x = 2$. **3**

Question 5 *(15 marks)* **Marks**

(a) A bicycle track with radius r is banked at an angle of α 7

to the horizontal so that a cyclist travelling at a certain speed V

will experience no sideways friction when riding around it.

(i) By resolving forces (either vertically and horizontally or parallel and perpendicular to the plane) show that

$$\tan\alpha = \frac{v^2}{rg}.$$

(ii) Find the value of α to the nearest degree if the speed required to experience no sideways friction on this track is $63km/h$, the radius of the track is 50 metres and $g = 9.8m/s^2$.

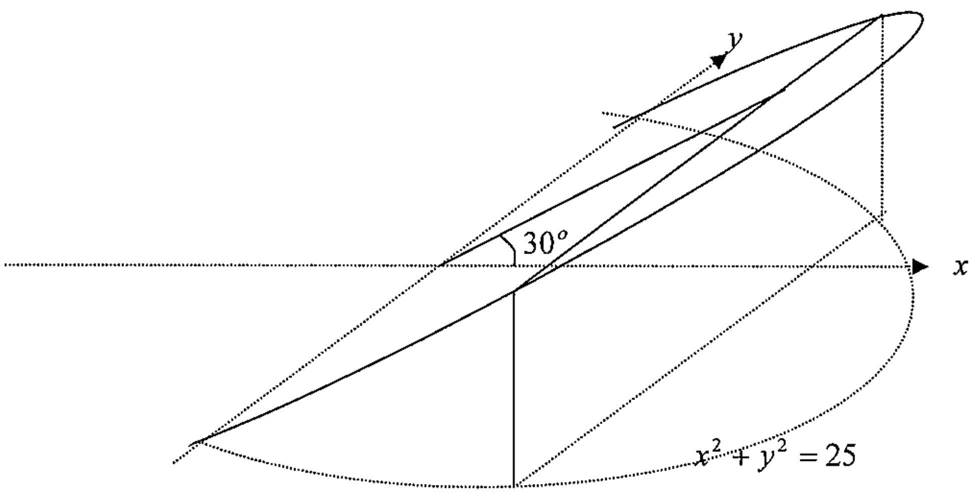
(iii) The maximum force that friction with the track can produce when the cyclist is travelling faster or slower than the optimum speed of $63km/h$ is $0.2\times$ the normal force. Find the minimum speed that the cyclist can travel at on the track before they start to slip down the track. (Use the results from parts (i) and (ii).)

Question (5) continues on the next page!

Question 5 (continued) **Marks**

(b) The area enclosed by the circle $x^2 + y^2 = 25$ and the y axis forms the base for a wedge shaped solid with its top at an angle of 30^o to the horizontal. 4

(See diagram.)



(i) Show that the area of each rectangular cross-section of the wedge which is perpendicular to the x axis is given by

$$\text{Area } = \frac{2x\sqrt{3}\sqrt{25-x^2}}{3}.$$

(ii) Find the volume of the wedge.

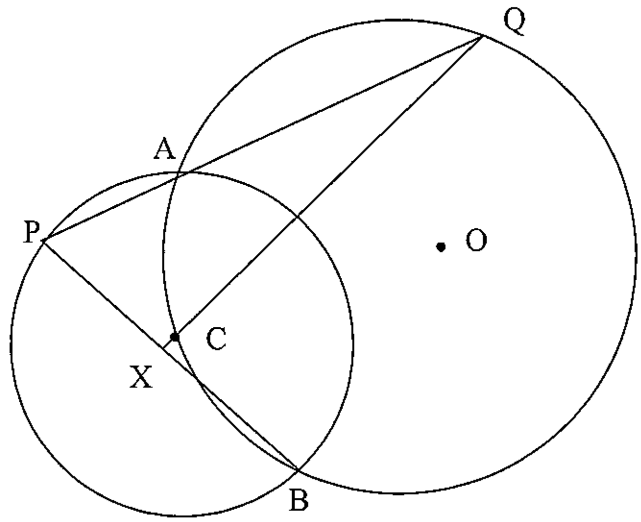
(c) A standard deck of cards consists of four of each kind (Ace, King, Queen etc.) and 13 of each suite (Spades, hearts, clubs and diamonds). A poker hand consists of 5 cards selected from this deck. 4

(i) How many hands would contain the Ace of clubs, the Ace of diamonds and no other pairs of like cards? (i.e. cards of the same kind.)

(ii) What is the probability that a poker hand will contain one pair of like cards?

Question 6 *(15 Marks)* **Marks**

(a) Two circles with centres O and C intersect at A and B. C lies on the circumference of the circle with centre O (see diagram.) **3**



A straight line PQ is drawn through A and QC Produced meets PB at X. Copy the diagram showing all of this information and prove that PX is perpendicular to CX and PX =XB.

(b) For the curve $x^2 + y^2 - xy = 48$ **8**

(i) Show that $\dfrac{dy}{dx} = \dfrac{y-2x}{2y-x}$

(ii) Find the points on the curve where the gradient is zero.

(iii) Find the points on the curve where the gradient is undefined.

(iv) Find the x and y intercepts and sketch the graph of $x^2 + y^2 - xy = 48$ showing all of these features.

Question (6) continues on the next page!

Question (6) (continued) **Marks**

(c) A shell (that is, one of the explosive kind the army uses!) is projected vertically upwards from the ground at an initial velocity of Um/s . It experiences air resistance which is proportional to the square of its velocity and gravity which is equal to $10m/s^2$ for each kilogram of its mass. **4**

(i) Show that while the shell is rising its position is given by $x = \frac{-1}{2k}\ln\left(\frac{10+kv^2}{10+kU^2}\right)$ where k is the constant for the air resistance.

(ii) Find U, the velocity at which the shell was launched, if it reaches a maximum height height of 500 metres and $k = 0.0004$.

Exam continues on the next page!

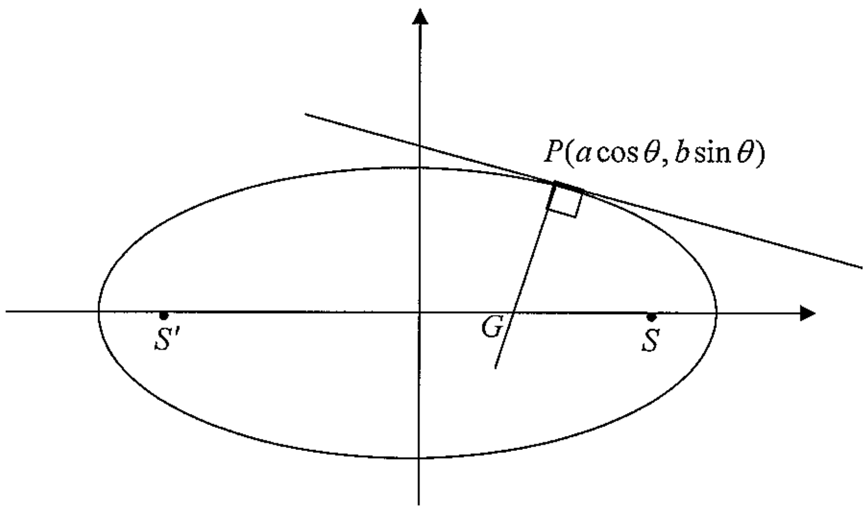
Question (7) *(15 marks)* **Marks**

(a) P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse **7**

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The normal at P intersects with the

x axis at G. S and S' are the foci of the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



(i) Show that the equation of the normal to the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at P is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$.

(ii) Show that the co-ordinates of G are $(ae^2\cos\theta, 0)$.

(iii) Show that $(PG)^2=(1-e^2)\times PS\times PS'$

Question (7) continues on the next page!

Question (7) (continued) **Marks**

(b)If w is the non real seventh root of 1 with the smallest positive argument: **8**

(i) Show that $w = cis\frac{2\pi}{7}$.

(ii) Show that $w^6 + w^5 + w^4 + w^3 + w^2 + w + 1 = 0$.

(iii) Show that $w + \frac{1}{w} = 2\cos\frac{2\pi}{7}$. Hence or otherwise show that $\cos\frac{2\pi}{7} - \cos\frac{\pi}{7} - \cos\frac{3\pi}{7} = -\frac{1}{2}$

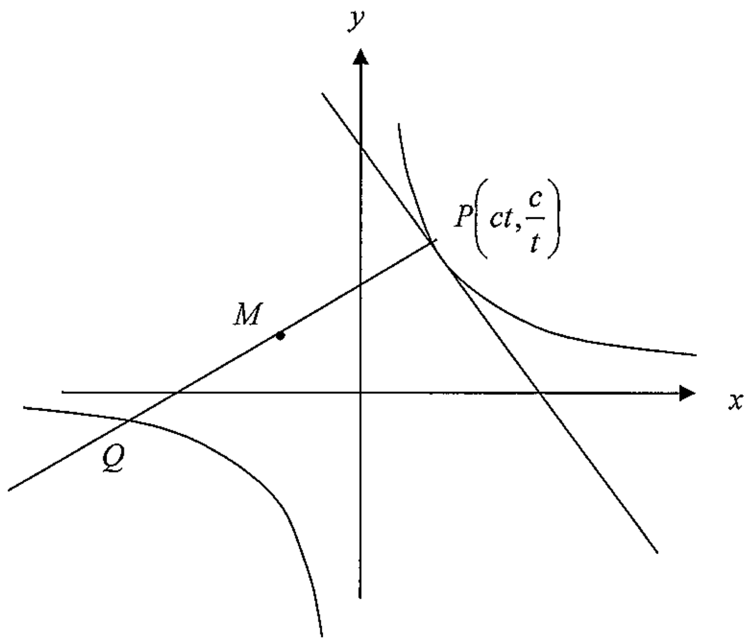
(iv) Using DeMoivre's theorem or otherwise, show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Hence or otherwise show that $\cos\frac{\pi}{7}$ is a root of the equation $8x^3 - 4x^2 - 4x + 1 = 0$.

(Do NOT attempt to solve this equation!)

Exam continues on the next page!

Question 8 *(15 marks)* **Marks**

(a) The normal to the rectangular hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ intersects with the hyperbola again at Q. The midpoint of the line PQ is M. (See diagram.) **8**



(i) Show that the equation of the normal to the hyperbola at P is $t^3x - ty = c(t^4 - 1)$

(ii) Prove that the co-ordinates of Q are $\left(-\frac{c}{t^3}, -ct^3\right)$

(iii) Show that the co-ordinates of M are $\left(\frac{c(t^4-1)}{2t^3}, -\frac{c(t^4-1)}{2t}\right)$.

(iv) Find the Cartesian equation for the locus of M.

(b) The equation $x^2 - x + 1 = 0$ has roots α and β and $A_n = \alpha^n + \beta^n$ for all $n \geq 1$. **7**

(i) Without solving the equation, show that $A_1 = 1$, $A_2 = -1$ and $A_n = A_{n-1} - A_{n-2}$ for all positive integers n.

(ii) Hence use induction to show that $A_n = 2\cos\frac{n\pi}{3}$ for all integers $n \geq 1$.

End of paper!!!

Extension 2 Trial Paper '09

Solutions & Marking Scheme   Muley GHS.

Q. (1)(a) $\displaystyle\int_1^e \frac{e^x}{1+e^{2x}}\,dx$ Let $u = e^x$. $x=e,\ u=e^e$.

$du = e^x\,dx$ $x=1\ \ u=e^1=e$

$$= \int_{u=e}^{u=e^e} \frac{1}{1+u^2}\,du$$

$$= \left[\tan^{-1}(u)\right]_e^{e^e}$$

3

$$= \tan^{-1}(e^e) - \tan^{-1}(e)$$

$$= 0.2866\ [4DP].$$

(b) $\displaystyle\int \frac{1}{3+\cos x}\,dx$ Let $t = \tan(\tfrac{1}{2}x)$

$dx = \dfrac{2}{t^2+1}\,dt$

$$= \int \frac{1}{3+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{t^2+1}\,dt$$

3

$$= \int \frac{2}{3t^2+3+1-t^2}\,dt$$

$$= \int \frac{2}{2t^2+4}\,dt$$

$$= \int \frac{1}{t^2+2}\,dt$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{\tan(\frac{1}{2}x)}{\sqrt{2}}\right] + C$$

(1) (c) $\int x \sin 2x \, dx$ $\quad u = x \quad v = -\frac{1}{2}\cos 2x$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = \sin 2x$

By $\int u \cdot \frac{dv}{dx} \, dx = uv - \int v \cdot \frac{du}{dx} \, dx$ $\quad$ 2

$$\int x \sin 2x \, dx = -\frac{x}{2}\cos 2x - \int -\frac{1}{2}\cos 2x \, dx$$

$$= -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C.$$

(d) $\frac{22-5x}{(x+1)(x-2)^2} \equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$\therefore 22 - 5x \equiv A(x-2)^2 + B(x+1)(x-2) + C(x+1)$

Substituting in $x = 2$	Substituting in $x = -1$
$22 - 5 \times 2 = C \times 3$	$22 + 5 \equiv A \times (-3)^2$
$12 = 3C$	$27 = 9A$
$4 = C$	$3 = A$

Substituting in $C = 4, A = 3, x = 0$

$22 - 5 \times 0 = 3 \times (-2)^2 + B \times 1 \times -2 + C \times 1$

$22 = 16 - 2B$

$-3 = B$ $\quad$ 4

$\therefore \frac{22-5x}{(x+1)(x-2)^2} = \frac{3}{x+1} - \frac{3}{x-2} + \frac{4}{(x-2)^2}$

$\therefore \int \frac{22-5x}{(x+1)(x-2)^2} \, dx$

$= \int \frac{3}{x+1} - \frac{3}{x-2} + \frac{4}{(x-2)^2} \, dx$

$= 3\ln\left(\frac{x+1}{x-2}\right) - \frac{4}{x-2} + C.$

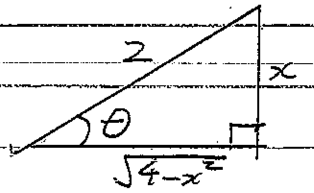
Ext. 2 Trial Paper Solutions '09 p.3

Q.(1)(e) $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,.dx$ Let $x = 2\sin\theta$.
$dx = 2\cos\theta . d\theta$

$= \displaystyle\int \frac{1}{4\sin^2\theta . 2\cos\theta} . 2\cos\theta . d\theta$ 1 for $2\cos\theta$ sub.

3

$= \dfrac{1}{4}\displaystyle\int \operatorname{cosec}^2\theta . d\theta$



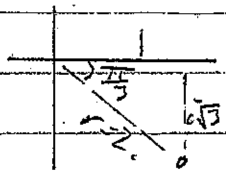
$= -\dfrac{1}{4}\cot\theta + C$ 1 for $\cot\theta$ 1 for converting back to x

$= -\dfrac{\sqrt{4-x^2}}{4x} + C$. $\Rightarrow$ Note: Would also pay $-\frac{1}{4}\cot\left[\sin^{-1}\left(\frac{x}{2}\right)\right] + C$

Q.(2)(a)(i) $\bar{z}w$
$= (1-3i)(2-i)$ 1 for $\bar{z}$
$= 2-i-6i-3$
$= -1-7i$ $\underline{2}$

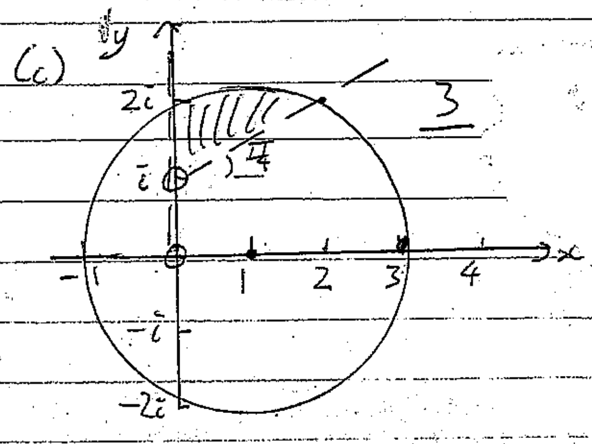
(ii) $\dfrac{z}{w}$
$= \dfrac{1+3i}{2-i} \times \dfrac{(2+i)}{(2+i)}$ 1
$= \dfrac{-1+7i}{5}$ MUST get to answer!

(b) $z = 1 - i\sqrt{3}$
$= 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$ 1



$\therefore z^5 = 2^5\operatorname{cis}\left(\frac{-5\pi}{3}\right)$
$= 32\operatorname{cis}\left(\frac{-5\pi}{3}\right)$ 1
$= 32\left[\frac{1}{2} + \frac{i\sqrt{3}}{2}\right]$ 3
$z^5 = 16 + 16i\sqrt{3}$ 1

(c)



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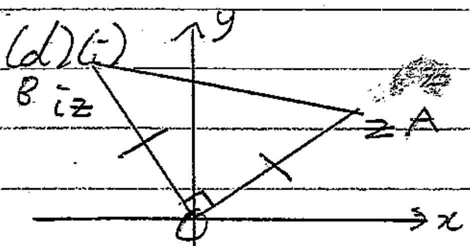
(d)(i)



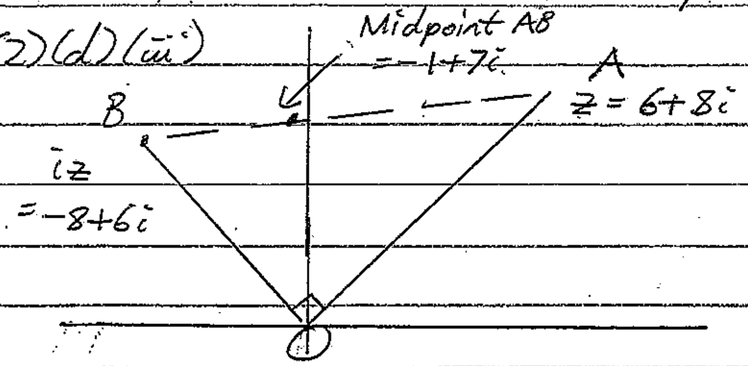
1

(ii) Distance $OA = |z|$.
Distance $OB = |iz| = |z|$
Distance $AB = |iz - z|$

By Pythagoras' Theorem: 2
$c^2 = a^2 + b^2$
$|iz - z|^2 = |z|^2 + |z|^2$
$|iz - z|^2 = 2|z|^2$

Extension 2 Trial Paper Solutions '09 (p. 4)

Q. (2)(d)(iii)



As $\angle AOB = 90^\circ$

AB must be a diameter of circle with O also at circumference [$\angle$ in semicircle $= 90^\circ$].

Hence circle centre must be midpoint of AB

$= -1 + 7i$ [Let this point $= H$]

& radius $= OH$

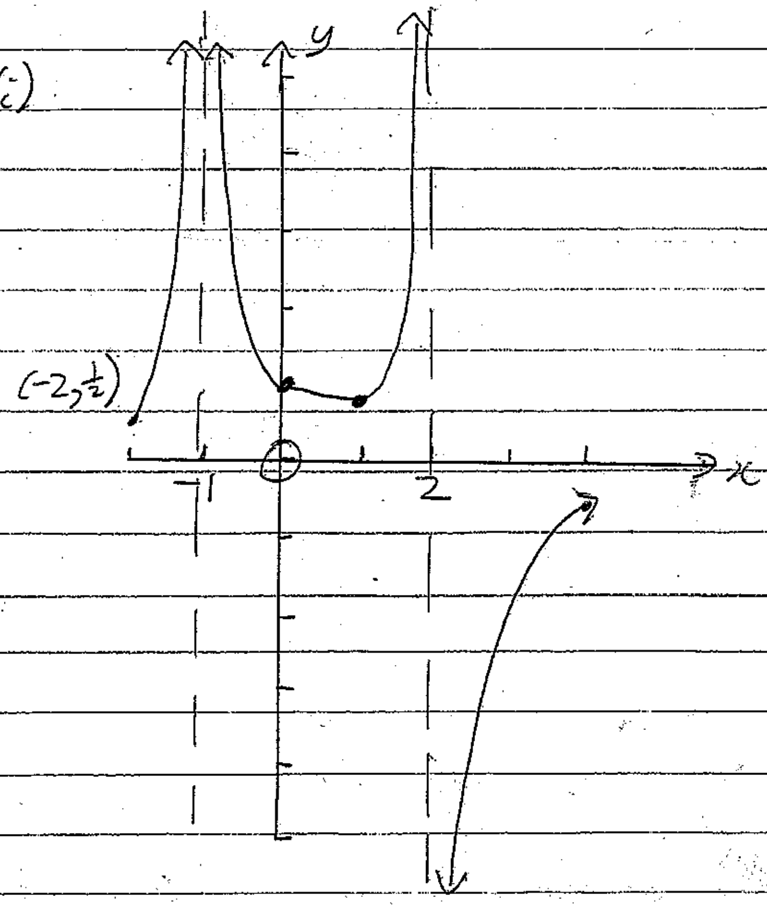
$= \sqrt{1^2 + 7^2}$

$= \sqrt{50}$.

3

$\therefore$ Circle with O, A & B on circumference has centre $-1+7i$ & radius $\sqrt{50}$.

Q. (3)(a)(i)



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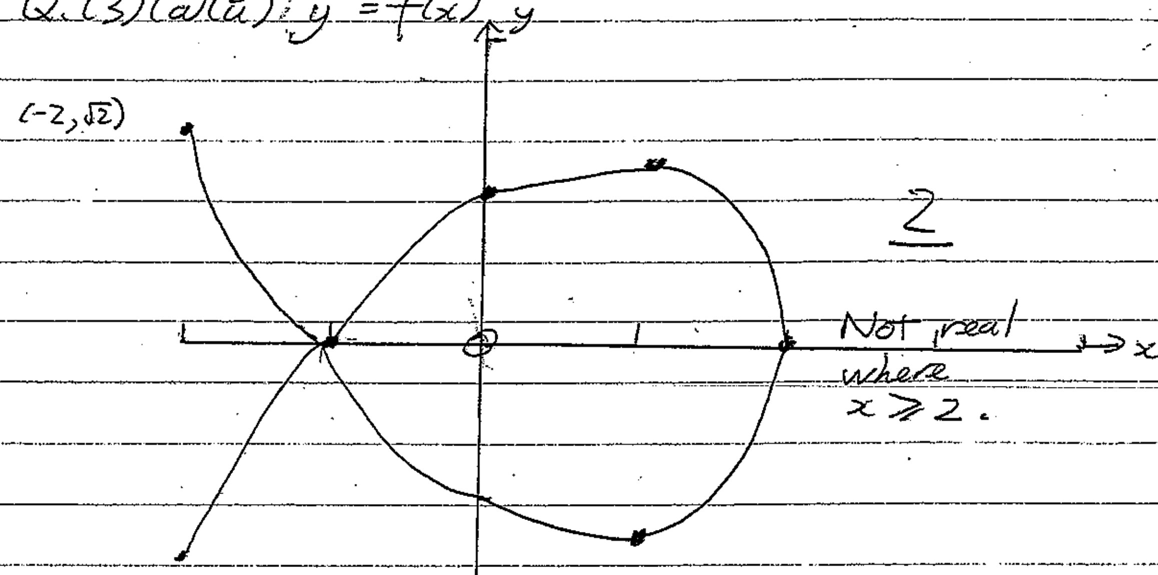
Q.(3)(a)(ii) $y^2 = f(x)$



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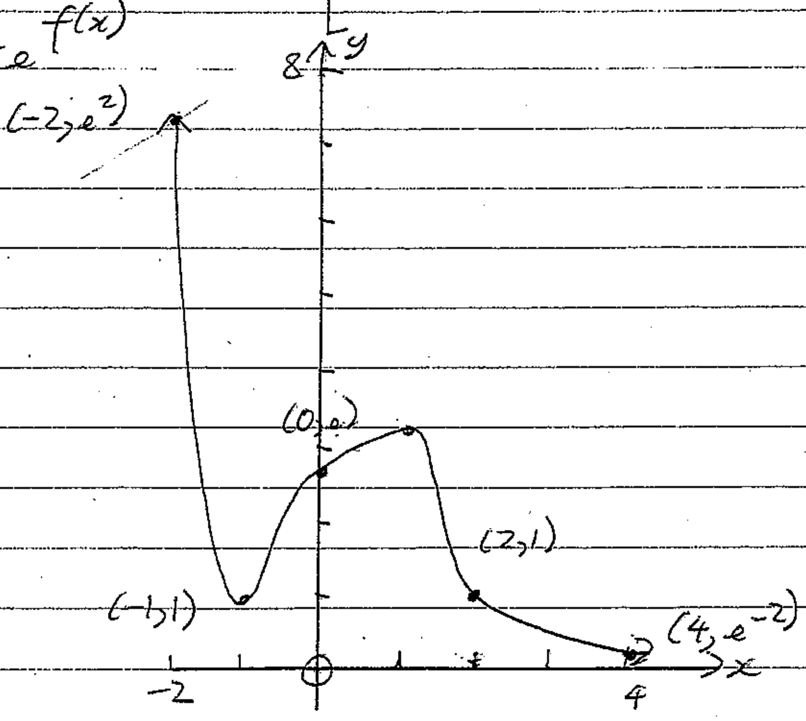
(iii) $y = e^{f(x)}$



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Ext. 2 '09 Trial Paper p. 6

Solutions & Marking Scheme

Q. (3)(b) (i) $e^2 = 1 + \frac{b^2}{a^2}$

$= 1 + \frac{16}{9}$

$= \frac{25}{9}$ 1

$e = \frac{5}{3}$.

(ii) Centre of hyperbola $= (1, -2)$

$ae = 3 \times \frac{5}{3} = 5$. So foci $= (6, -2)$ & $(-4, -2)$. 2

(iii) Directrices: $\frac{a}{e} = 3 \times \frac{3}{5} = \frac{9}{5}$.

So directrices are $x = 1 - \frac{9}{5}$ & $x = 1 + \frac{9}{5}$ 2

$= -\frac{4}{5}$ $= \frac{14}{5}$.

Directrices are the vertical lines $x = -\frac{4}{5}$ & $x = \frac{14}{5}$.

(iv) Asymptotes are $y = \pm \frac{b}{a}x$ but passing through $(1, -2)$

$y = \pm \frac{4}{3}x$. 1

By $y - y_1 = m(x - x_1)$

$y + 2 = \frac{4}{3}(x - 1)$

$3y + 6 = 4x - 4$. 1

$0 = 4x - 3y - 10$.

By $y - y_1 = m(x - x_1)$

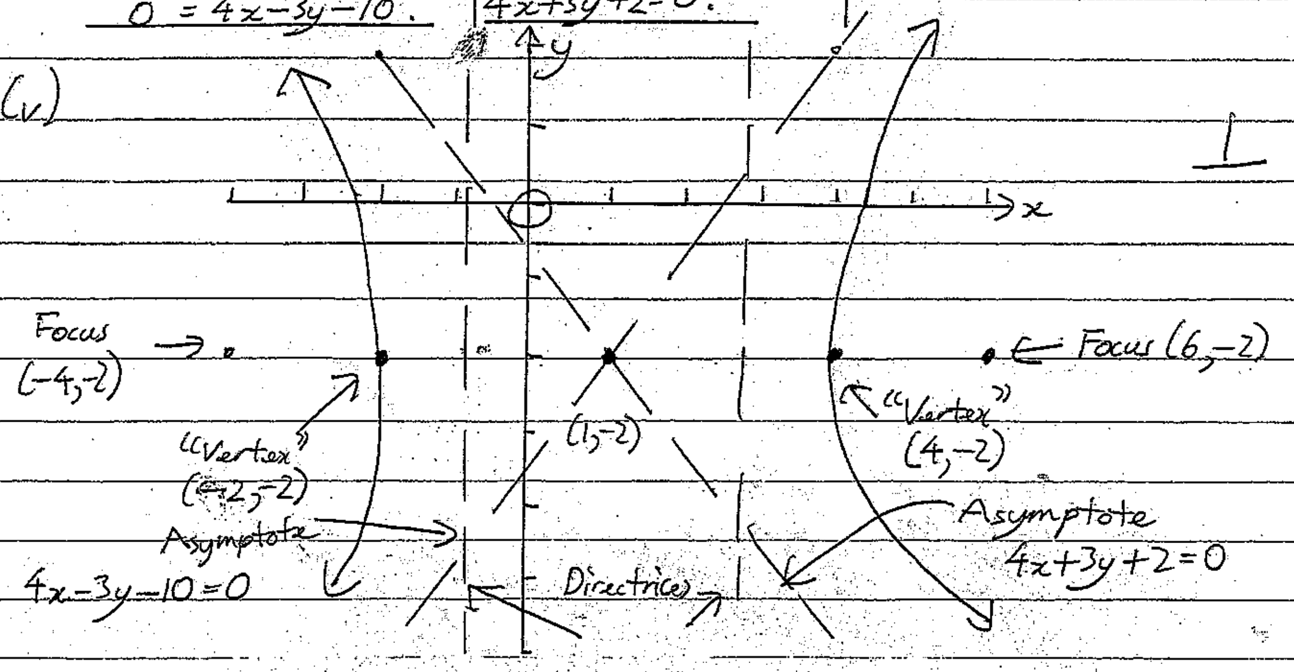
$y + 2 = -\frac{4}{3}(x - 1)$

$3y + 6 = -4x + 4$. 1

$4x + 3y + 2 = 0$.

The asymptotes are $4x - 3y - 10 = 0$ and $4x + 3y + 2 = 0$ 3

(v) 1



Q.(4)(a)(i) $\alpha^2+\beta^2+\delta^2$

$= (\alpha+\beta+\delta)^2 - 2(\alpha\beta+\alpha\delta+\beta\delta)$

$= \left(\frac{2}{3}\right)^2 - 2\times\frac{5}{3}$

Notes:

$\alpha+\beta+\delta = \frac{-b}{a} = \frac{2}{3}.$

$\alpha\beta+\alpha\delta+\beta\delta = \frac{c}{a} = \frac{5}{3}.$

$\alpha^2+\beta^2+\delta^2 = -\frac{26}{9}$ $\quad \underline{2}$

(ii) As α, β, δ are roots of $3x^3-2x^2+5x+1=0$

$3\alpha^3-2\alpha^2+5\alpha+1=0$

$3\beta^3-2\beta^2+5\beta+1=0$

$3\delta^3-2\delta^2+5\delta+1=0$

$\therefore 3(\alpha^3+\beta^3+\delta^3)-2(\alpha^2+\beta^2+\delta^2)+5(\alpha+\beta+\delta)+3=0$

$3(\alpha^3+\beta^3+\delta^3)-2\times\frac{-26}{9}+5\times\frac{2}{3}+3=0$

$3(\alpha^3+\beta^3+\delta^3) = \frac{-109}{9}$

$\alpha^3+\beta^3+\delta^3 = \frac{-109}{27}$ $\quad \underline{2}$

(iii) From Part (i) As $\alpha^2+\beta^2+\delta^2<0$ at least one of α, β, δ must not be real. However, as the co-efficients of $P(x)$ are all real the CONJUGATE of this root must also be a root. $\underline{2}$ Hence, $P(x)$ has two non-real roots.

(iv) Letting $y=x^2$, $x=\sqrt{y}$

So $P(x)=3x^3-2x^2+5x+1=0$ becomes

$3y\sqrt{y}-2y+5\sqrt{y}+1=0$

$3y\sqrt{y}+5\sqrt{y} = 2y-1$

$\sqrt{y}(3y+5) = 2y-1$

Squaring BS: $y(9y^2+30y+25) = 4y^2-4y+1$ $\quad \underline{2}$

$9y^3+26y^2+29y-1=0$

Replacing y with x, polynomial with roots $\alpha^2, \beta^2, \delta^2 = \underline{9x^3+26x^2+29x-1.}$

[not $x=\sqrt{y}$].

Q.(4)(b) $I_n = \int \cos^n x . dx$

$= \int \cos^{n-1} x . \cos x . dx$

$u = \cos^{n-1} x$	$v = \sin x$
$\frac{du}{dx} = -\sin x . (n-1) . \cos^{n-2} x$	$\frac{dv}{dx} = \cos x$

By $\int u \frac{dv}{dx} . dx = uv - \int v \frac{du}{dx} . dx$

$I_n = \int \cos^n x . dx \quad = \cos^{n-1} x . \sin x - \int \sin x . -\sin x . (n-1) \cos^{n-2} x . dx$

$= \cos^{n-1} x . \sin x + \int (n-1) \cos^{n-2} x . (1 - \cos^2 x) . dx$

$\int \cos^n x . dx = \cos^{n-1} x . \sin x + (n-1) \int \cos^{n-2} x . dx - (n-1) \int \cos^n x . dx$

i.e. $I_n = \cos^{n-1} x . \sin x + (n-1) I_{n-2} - (n-1) I_n$

$n I_n = \cos^{n-1} x . \sin x + (n-1) I_{n-2}$

$\therefore I_n = \frac{\cos^{n-1} x . \sin x}{n} + \frac{(n-1)}{n} I_{n-2}$.

$\therefore$ Introducing limits of $\frac{\pi}{2}$ & 0

$I_0 = \int_0^{\frac{\pi}{2}} 1 . dx = \frac{\pi}{2}$

$I_2 = \left[\frac{\cos x \sin x}{2}\right]_0^{\frac{\pi}{2}} + \frac{1}{2} I_0$

$I_4 = \left[\frac{\cos^3 x \sin x}{4}\right]_0^{\frac{\pi}{2}} + \frac{3}{4} I_2$

$I_6 = \left[\frac{\cos^5 x \sin x}{6}\right]_0^{\frac{\pi}{2}} + \frac{5}{6} I_4$

$I_8 = \left[\frac{\cos^7 x \sin x}{8}\right]_0^{\frac{\pi}{2}} + \frac{7}{8} I_6$

and $I_{10} = \left[\frac{\cos^9 x \sin x}{10}\right]_0^{\frac{\pi}{2}} + \frac{9}{10} I_8$.

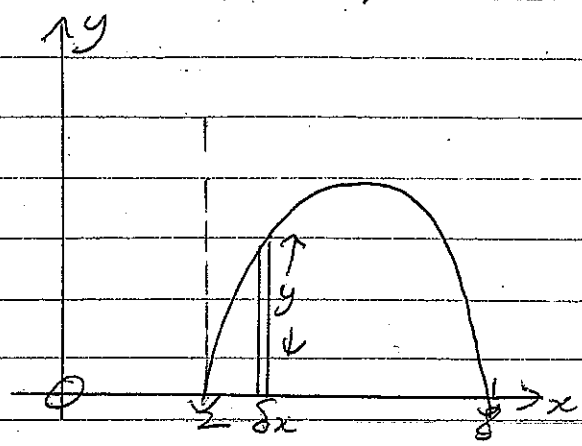
As limits are from $\frac{\pi}{2}$ to 0 & $\cos \frac{\pi}{2} = 0$, $\sin 0 = 0$ all $\cos x \times \sin x$ terms can be ignored.

$\therefore \int_0^{\frac{\pi}{2}} \cos^{10} x . dx = \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$

$= \frac{63\pi}{256}$.

4

Q.(4)(i) $y = 10x - x^2 - 16$



Finding x intercepts

$0 = 10x - x^2 - 16$

$0 = x^2 - 10x + 16$

$0 = (x-8)(x-2)$

$x = 2$ or 8

→ As graph is a parabola it is upside down.

Outer radius $r_2 = (x-2) + \delta x$.

Inner radius $r_1 = x - 2$.

3

$$\delta V = \pi y\left[\left[(x-2)+\delta x\right]^2 - (x-2)^2\right]$$

$$= 2\pi y(x-2).\delta x.$$

$$\therefore V = 2\pi \lim_{\delta x \to 0} \sum_{x=2}^{x=8} (x-2)y.\delta x.$$

$$= 2\pi \lim_{\delta x \to 0} \sum_{x=2}^{x=8} (x-2)(10x - x^2 - 16).\delta x$$

Letting $\delta x \to 0$

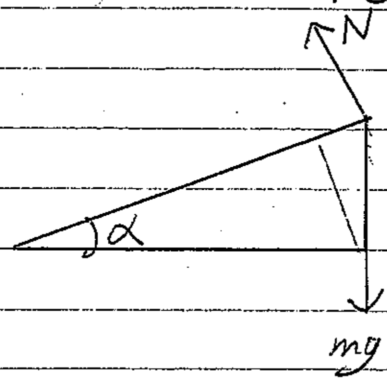
$$V = -2\pi \int_2^8 x^3 - 12x^2 + 36x - 32.dx$$

$$= -2\pi \left[\frac{1}{4}x^4 - 4x^3 + 18x^2 - 32x\right]_2^8$$

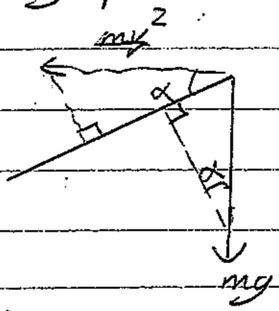
$$= -2\pi\left[-128 - -20\right]$$

$$= \underline{216\pi \text{ cubic units.}}$$

Q.(5)(a)(i)



Resolving parallel to plane:

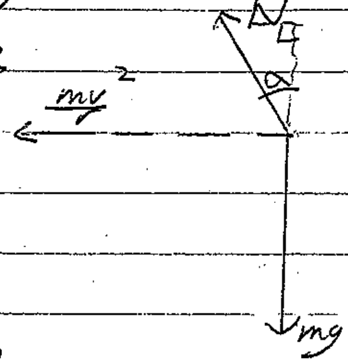


$mg\sin\alpha = \frac{mv^2}{r}\cos\alpha$

Dividing BS by $mg\cos\alpha$

$\tan\alpha = \frac{v^2}{rg}$.

OR



2

OR

Resolving horizontally:

$N\sin\alpha = \frac{mv^2}{r}$ (1)

Resolving vertically:

$N\cos\alpha = mg$ (2)

Dividing (1) by (2)

$\tan\alpha = \frac{v^2}{rg}$.

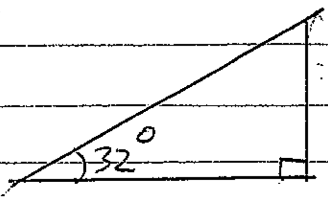
(ii) Finding α: $g = 9.8\,m/s$. $v = 63\,km/h = 17.5\,m/s$. $r = 50$ metres.

$\tan\alpha = \frac{v^2}{rg}$

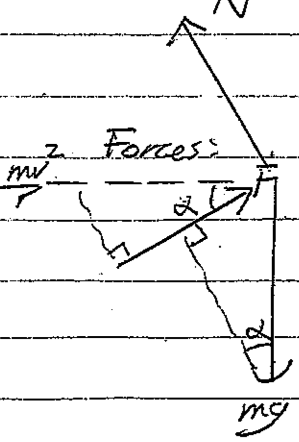
$= \frac{17.5^2}{50\times 9.8}$

$\alpha = 32°$. 1

(iii)



Forces:



Parallel to plane: 4

$mg\sin\alpha - F = \frac{mv^2}{r}\cos\alpha$

As $F = 0.2N$

$mg\sin\alpha - 0.2N = \frac{mv^2}{r}\cos\alpha$ (1)

Perpendicular to planes

$N - mg\cos\alpha = \frac{mv^2}{r}\sin\alpha$ (2)

PTO →

Q.(5)(a)(iii) [continued]: (1) from previous page $\times 5 = (3)$

$$5mg\sin\alpha - N = \frac{5mv^2}{r}\cos\alpha \quad (3)$$

$+$

$$-mg\cos\alpha + N = \frac{mv^2}{r}\sin\alpha \quad (2)$$

$$5mg\sin\alpha - mg\cos\alpha = \frac{5mv^2}{r}\cos\alpha + \frac{mv^2}{r}\sin\alpha$$

$\times$ BS by $\frac{r}{m}$

$$r[5g\sin\alpha - g\cos\alpha] = 5v^2\cos\alpha + v^2\sin\alpha$$

$$\frac{r[5g\sin\alpha - g\cos\alpha]}{(5\cos\alpha + \sin\alpha)} = v^2$$

As $r = 50$, $g = 9{\cdot}8$, $\alpha = 32^\circ$.

$$\frac{50\times[5\times 9{\cdot}8\times\sin 32^\circ - 9{\cdot}8\cos 32^\circ]}{(5\cos 32^\circ + \sin 32^\circ)} = v^2$$

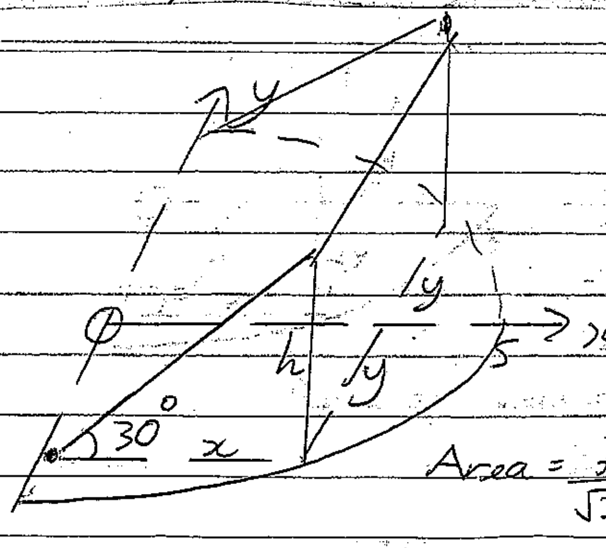
$13{\cdot}6\ldots$ m/s $= v$

Minimum speed before falling down track

$\doteqdot 13{\cdot}6$ m/s

$\doteqdot 49$ km/h.

(b)(i)



Height of each rectangle

$= x\tan 30^\circ$

$= \frac{x}{\sqrt{3}}$

Breadth $= 2y$

$= 2\sqrt{25-x^2}$

$$\text{Area} = \frac{x}{\sqrt{3}} \times 2\sqrt{25-x^2} \quad \frac{\times\sqrt{3}}{\times\sqrt{3}}$$

$$= \frac{2x\sqrt{3}\sqrt{25-x^2}}{3}$$

$\underline{2}$

Q. (5) (b) [continued]:

(ii) $\delta V = \dfrac{2x\sqrt{3}\sqrt{25-x^2}}{3} \cdot \delta x$

$\therefore V = \lim_{\delta x \to 0} \sum_{x=0}^{x=5} \dfrac{2x\sqrt{3}}{3} \sqrt{25-x^2} \cdot dx$

$= \dfrac{-\sqrt{3}}{3} \int_0^5 -2x\sqrt{25-x^2} \cdot dx$ Letting $u = 25 - x^2$, $du = -2x \cdot dx$

$= -\dfrac{\sqrt{3}}{3} \int_{u=25}^{u=0} \sqrt{u} \cdot du$

$= \dfrac{\sqrt{3}}{3} \left[\dfrac{2}{3} u\sqrt{u} \right]_0^{25}$ → Taking out minus sign & swapping limits!

$= \dfrac{2\sqrt{3}}{9} \left[25 \times 5 - 0 \right]$

$= \dfrac{250\sqrt{3}}{9}$ c. u. (cubic units).

2

(c) (i) Ace Clubs Ace Diamonds $\dfrac{\overset{[\text{4 aces out}]}{48} \times \overset{[\text{4 Aces \& 4 another kind out}]}{44} \times \overset{[\text{4 Aces 4 of 2 other kinds out}]}{40}}{3!}$ → because these 3 cards could be selected in any order

2

$= 14\,080$ hands.

(ii) Number of hands with one pair of like cards

$= {}^4C_2 \times 13 \times 14\,080 = 1\,098\,240$ pairs.

↑ Pairs per kind ↑ 13 kinds

~~Total~~ number of hands $= {}^{52}C_5$

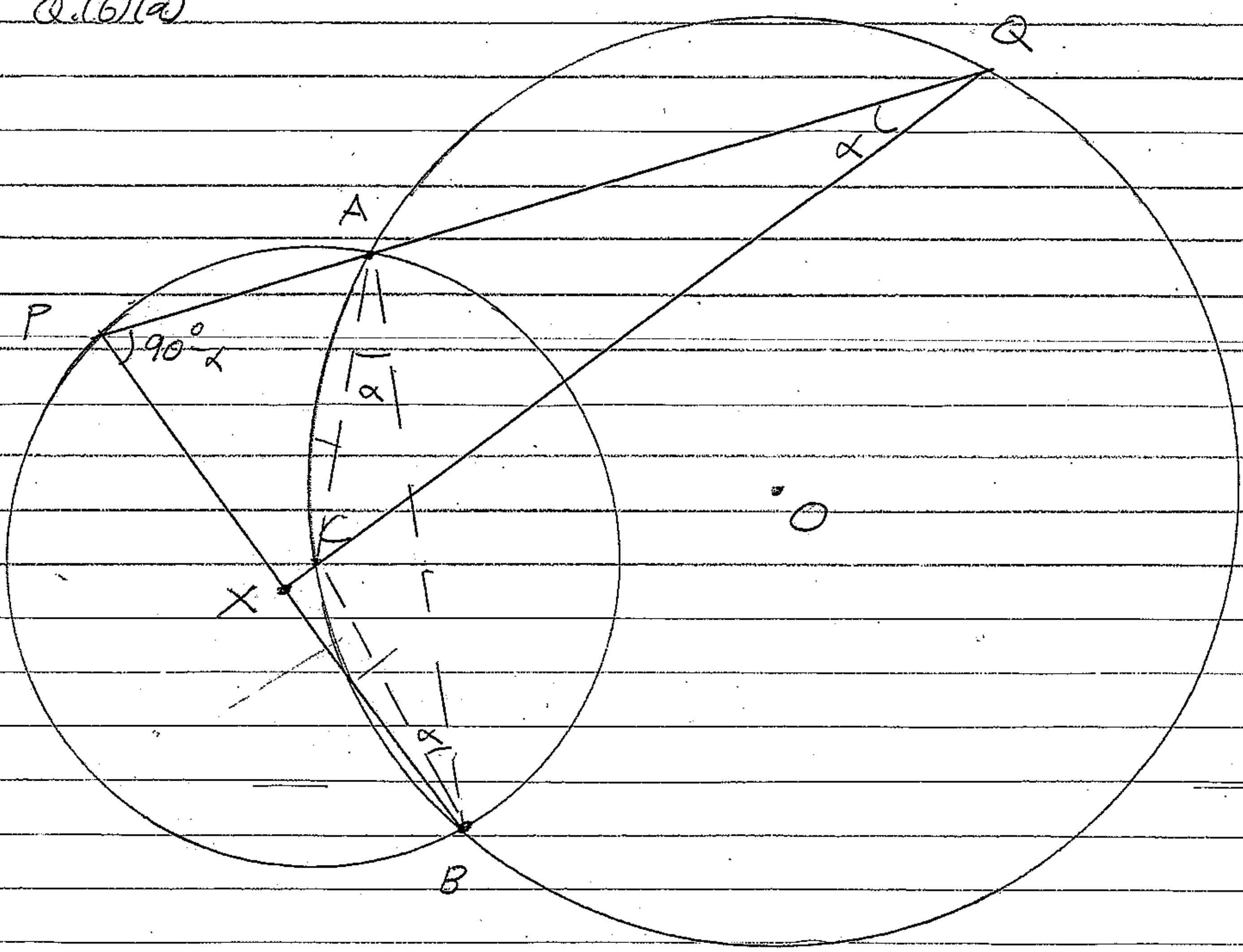
$= 2\,598\,960$.

Probability [one pair] $= \dfrac{1\,098\,240}{2\,598\,960}$

$= \dfrac{352}{833}$.

2

Q.(6)(a)



Join AC, BC, AB

Let $\angle AQC = \alpha$

$\angle ABC = \alpha$ [$\angle$'s on same arc AC of circle centre O].

$AC = BC$ [radii of circle centre C].

$\angle CAB = \alpha$ [$\angle$'s opposite = sides in isosceles $\triangle ABC$].

$\angle ACB = 180° - 2\alpha$ [$\angle$ sum $\triangle ABC$].

$\angle APB = 90° - \alpha$ [$\angle$ at circumference is $\frac{1}{2}$ $\angle$ at centre of circle on same arc].

$\therefore \angle PXQ = 90°$ [$\angle$ sum $\triangle PXQ$].

$\therefore PX \perp XQ$ 3

Also $PX = XB$ [line through centre of circle C $\perp$ chord bisects chord].

Q. (6)(b)

(i) $x^2+y^2-xy=48$

$$2x+2y\frac{dy}{dx}-\left(y+x\frac{dy}{dx}\right)=48$$

$$\frac{dy}{dx}(2y-x)=y-2x$$

$$\frac{dy}{dx}=\frac{y-2x}{2y-x}$$

2

(ii) $\frac{dy}{dx}=0$ where $y-2x=0$

i.e. $y=2x$

Sub. back in $x^2+y^2-xy=48$

$x^2+(2x)^2-x\times 2x=48$

$3x^2=48 \Rightarrow x=\pm 4.$

As $y=2x$, $y=\pm 8$

2

$\frac{dy}{dx}=0$ at $(-4,-8)$ & $(4,8)$

(iii) $\frac{dy}{dx}$ is undefined where $2y-x=0$

i.e. $x=2y$.

Sub. back in $x^2+y^2-xy=48$

$(2y)^2+y^2-2y\times y=48$

$3y^2=48$

2

$y=\pm 4 \Rightarrow x=\pm 8$.

$\frac{dy}{dx}$ is undefined at $(-8,-4)$ and $(8,4)$.

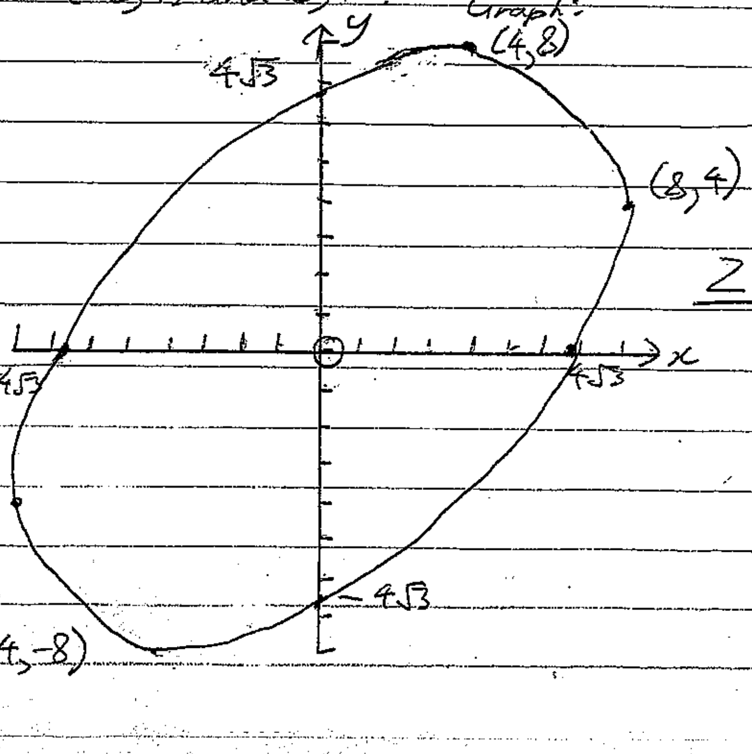
(iv) Intercepts: x intercepts $(y=0)$

$x^2=48 \Rightarrow x=\sqrt{48}=4\sqrt{3}$.

y intercepts:

$y^2=48 \Rightarrow y=\pm 4\sqrt{3}$.

Graph:



2

Q. (6)(c)(i)

$\downarrow mg = 10m \quad \downarrow mkv^2$

$$a = v\frac{dv}{dx} = -10 - kv^2$$

$$\frac{dv}{dx} = \frac{-(10+kv^2)}{v}$$

$$\frac{dx}{dv} = \frac{-v}{10+kv^2}$$

$$x = -\frac{1}{2k}\int \frac{2kv}{10+kv^2}\,dv$$

$$x = -\frac{1}{2k}\ln(10+kv^2) + C$$

As $v = U$ when $x = 0$

$$0 = -\frac{1}{2k}\ln(10+kU^2) + C$$

$$\therefore C = \frac{1}{2k}\ln(10+kU^2) \qquad \underline{3}$$

$$x = -\frac{1}{2k}\ln\left(\frac{10+kv^2}{10+kU^2}\right)$$

(ii) As $x = 500$ when $v = 0$ & $k = 0.0004$

$$500 = -\frac{1}{2\times 0.0004}\ln\left(\frac{10}{10+0.0004U^2}\right)$$

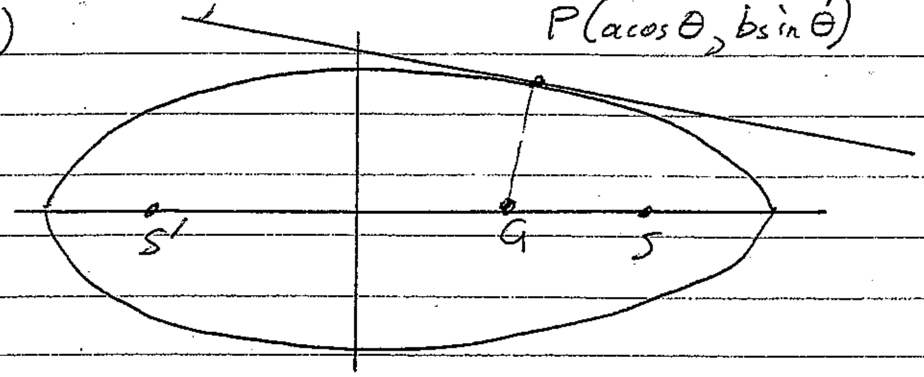
$$0.4 = \ln\left(\frac{10+0.0004U^2}{10}\right)$$

$$e^{0.4} = \frac{10+0.0004U^2}{10}$$

$$\frac{10e^{0.4}-10}{0.0004} = U^2$$

$$\underline{U = 110.885\ldots \text{ m/s.}} \qquad \underline{1}$$

Q. (7)(a)



(i) To find m of normal:

m of tangent: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\frac{2y}{b^2} \cdot \frac{dy}{dx} = \frac{-2x}{a^2}$$

$$\frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{b^2}{2y}$$

$$= \frac{-b^2x}{a^2y}.$$

$\therefore$ m of tangent at $(a\cos\theta, b\sin\theta)$

$$= \frac{-b^2 a\cos\theta}{a^2 b\sin\theta}$$

$$= \frac{-b\cos\theta}{a\sin\theta}$$

$\therefore$ m of normal at $(a\cos\theta, b\sin\theta)$

$$= \frac{a\sin\theta}{b\cos\theta}.$$

2

Equation of normal at $(a\cos\theta, b\sin\theta)$

By $y - y_1 = m(x - x_1)$

$$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}\left(x - a\cos\theta\right)$$

$\times$ BS by $b\cos\theta$.

$$by\cos\theta - b^2\sin\theta\cos\theta = a^2x\sin\theta - a^2\sin\theta\cos\theta$$

$$a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - by\cos\theta$$

Dividing BS by $\sin\theta\cos\theta$

$$a^2 - b^2 = \frac{ax}{\cos\theta} - \frac{by}{\sin\theta}.$$

Extension 2 Trial Paper p. 17

Q. (7)(a)(ii) G is x intercept of [y=0].

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

i.e. $$\frac{ax}{\cos\theta} = a^2 - b^2$$ 1

$$ax = (a^2 - b^2)\cos\theta.$$

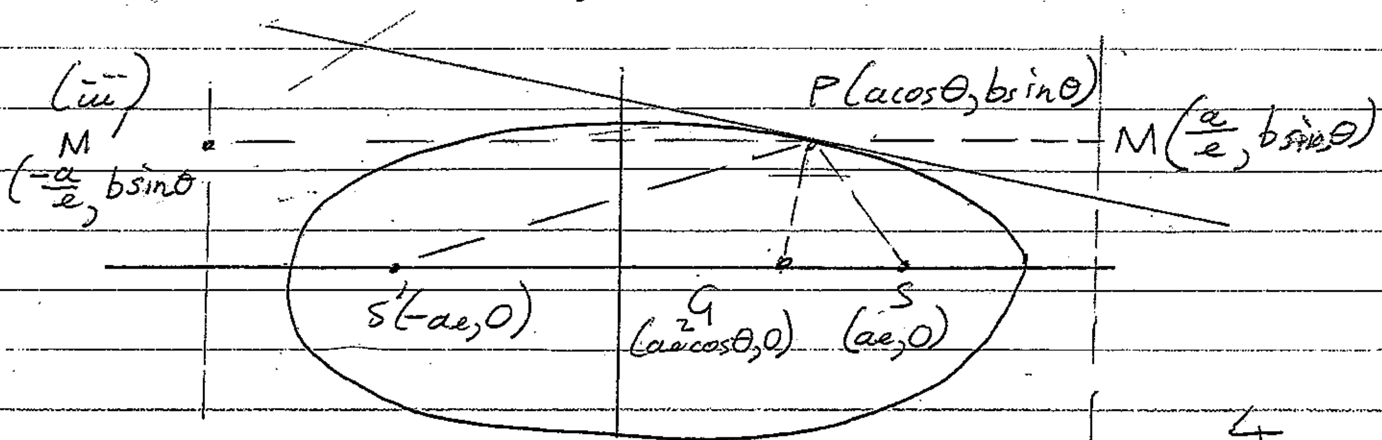
$$x = \left(a - \frac{b^2}{a}\right)\cos\theta$$

$$= a\times\left(1 - \frac{b^2}{a^2}\right)\cos\theta$$

$$x = ae^2\cos\theta \text{ as } e^2 = 1 - \frac{b^2}{a^2}.$$

$\therefore$ G is $(ae^2\cos\theta, 0)$.

(iii)



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By definition of ellipse:

$PS = e \times PM$

$= e\times\left(\frac{a}{e} - a\cos\theta\right)$

$= a - ae\cos\theta$

$= a(1 - e\cos\theta)$ 1

Similarly

$PS' = e \times PM'$

$= e\left(\frac{a}{e} + a\cos\theta\right)$

$= a + ae\cos\theta$

$= a(1 + e\cos\theta)$

$\therefore PS \times PS'$

$= a^2(1 - e^2\cos^2\theta)$

& $(1-e^2)\times PS \times PS'$

$= a^2(1-e^2)(1-e^2\cos^2\theta)$

$= b^2(1-e^2\cos^2\theta)$ as $b^2 = a^2(1-e^2)$

By Pythagoras' theorem [or distance formula]

$(PG)^2 = (a\cos\theta - ae^2\cos\theta)^2 + |b^2\sin^2\theta|$

$= a^2\cos^2\theta(1-e^2)^2 + b^2\sin^2\theta$

$= a^2\cos^2\theta(1-e^2)^2 + b^2(1-\cos^2\theta)$

$= a^2\cos^2\theta(1-e^2)^2 + b^2 - b^2\cos^2\theta$

$= \frac{b^2\cos^2\theta}{1-e^2}(1-e^2)^2 + b^2 - b^2\cos^2\theta$

$= b^2\cos^2\theta(1-e^2) + b^2 - b^2\cos^2\theta$

$= b^2\cos^2\theta - b^2e^2\cos^2\theta + b^2 - b^2\cos^2\theta$

$= b^2(1 - e^2\cos^2\theta)$

$=$ LHS.

Extension 2 Trial Solutions p. 18

Q.(7)(b)(i) Roots of $z^7 = 1$

are $cis\frac{2\pi}{7}, cis\frac{4\pi}{7}, cis\frac{6\pi}{7}, cis\frac{8\pi}{7}, cis\frac{10\pi}{7}, cis\frac{12\pi}{7}, cis 2\pi$ (1).

$\therefore$ Root [ω] with smallest positive argument $= cis\frac{2\pi}{7}$. 1

(ii) If $z^7 = 1$

$z^7 - 1 = 0$

$(z-1)(z^6+z^5+z^4+z^3+z^2+z+1) = 0$

As $\omega^7 = 1$ and $\omega \neq 1$ 1

$\omega^6+\omega^5+\omega^4+\omega^3+\omega^2+\omega+1 = 0$

Note: Alternative

- Show $\omega^2 = cis\frac{4\pi}{7}$ which is a root
- Show $\omega^3 = cis\frac{6\pi}{7}$ " "

By sum of roots:

$\alpha + \beta + \ldots = -\frac{b}{a}$

$1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6 = 0.$

(iii) $\omega + \frac{1}{\omega}$

$= \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} + \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$

$= 2\cos\frac{2\pi}{7}$

Note: $\frac{1}{\omega} = cis\left(-\frac{2\pi}{7}\right)$

$= \cos\left(-\frac{2\pi}{7}\right) + i\sin\left(-\frac{2\pi}{7}\right)$

$= \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}.$

As we know $\omega^6+\omega^5+\omega^4+\omega^3+\omega^2+\omega+1 = 0$

Dividing BS by ω^3 & rearranging [as $\omega \neq 0$ no problems].

$\omega^3+\frac{1}{\omega^3}+\omega^2+\frac{1}{\omega^2}+\omega+\frac{1}{\omega}+1 = 0$

[Using $\omega+\frac{1}{\omega} = 2\cos\frac{2\pi}{7}$ etc].

$2\cos\frac{6\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{2\pi}{7} + 1 = 0$

As $\cos\frac{6\pi}{7} = -\cos\frac{\pi}{7}$ and $\cos\frac{4\pi}{7} = -\cos\frac{3\pi}{7}$.

3

$2\cos\frac{2\pi}{7} - 2\cos\frac{\pi}{7} - 2\cos\frac{3\pi}{7} + 1 = 0$

$\cos\frac{2\pi}{7} - \cos\frac{\pi}{7} - \cos\frac{3\pi}{7} + \frac{1}{2} = 0$

$\cos\frac{2\pi}{7} - \cos\frac{\pi}{7} - \cos\frac{3\pi}{7} = -\frac{1}{2}.$

Q. (7)(b)(iv) By De Moivre's theorem

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = \cos 3\theta + i\sin 3\theta$$

Equating real parts: $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$$

$$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

[Note: You could do "or otherwise": $\cos 3\theta = \cos(2\theta+\theta)$ etc].

Letting $\theta = \frac{\pi}{7}$

$$\cos\frac{2\pi}{7} = \cos 2\theta = 2\cos^2\frac{\pi}{7} - 1.$$

$$\cos\frac{3\pi}{7} = \cos 3\theta = 4\cos^3\frac{\pi}{7} - 3\cos\frac{\pi}{7}.$$

As we know that

$\cos\frac{2\pi}{7} - \cos\frac{\pi}{7} - \cos\frac{3\pi}{7} = -\frac{1}{2}$ from Part (iii)].

$$\left(2\cos^2\frac{\pi}{7} - 1\right) - \cos\frac{\pi}{7} - \left(4\cos^3\frac{\pi}{7} - 3\cos\frac{\pi}{7}\right) = -\frac{1}{2}.$$

$$-4\cos^3\frac{\pi}{7} + 2\cos^2\frac{\pi}{7} + 2\cos\frac{\pi}{7} - 1 = -\frac{1}{2}.$$

$\times$ BS by -2

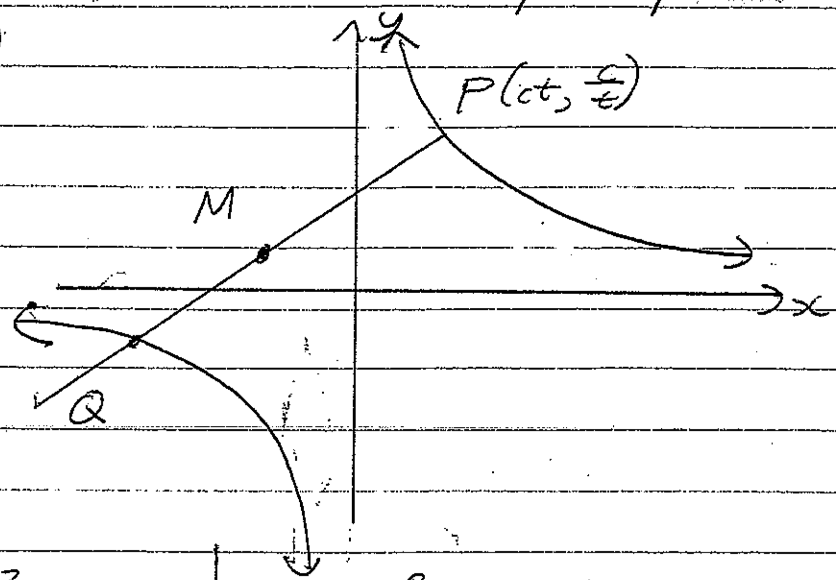
$\underline{3}$

$$8\cos^3\frac{\pi}{7} - 4\cos^2\frac{\pi}{7} - 4\cos\frac{\pi}{7} + 2 = 1.$$

$$8\cos^3\frac{\pi}{7} - 4\cos^2\frac{\pi}{7} - 4\cos\frac{\pi}{7} + 1 = 0$$

$\therefore \cos\frac{\pi}{7}$ is a root of $8x^3 - 4x^2 - 4x + 1 = 0.$

Q.(8)(a)



(i) $xy = c^2$

$y = \frac{c^2}{x}$

$\frac{dy}{dx} = -\frac{c^2}{x^2}$

At $P\left(ct, \frac{c}{t}\right)$

$\frac{dy}{dx} = \frac{-c^2}{c^2t^2}$

$= -\frac{1}{t^2}$

As m of tangent $= -\frac{1}{t^2}$

m of normal $= t^2$

By $y - y_1 = m(x - x_1)$

$y - \frac{c}{t} = t^2(x - ct)$

×BS by t.

$ty - c = t^3x - ct^4$

$ct^4 - c = t^3x - ty$

$\therefore t^3x - ty = c(t^4 - 1)$.

3

(ii) Q is other point of intersection with hyperbola:

$xy = c^2 \Rightarrow y = \frac{c^2}{x}$

Sub. in $t^3x - ty = c(t^4-1)$

$t^3x - t \times \frac{c^2}{x} = c(t^4-1)$

×BS by x.

$t^3x^2 - c^2t = cx(t^4-1)$

$\therefore t^3x^2 - cx(t^4-1) - c^2t = 0$ (1)

By quadratic formula, $x = \frac{c(t^4-1) \pm \sqrt{c^2(t^4-1)^2 - 4 \times t^3 \times -c^2t}}{2t^3}$

$= \frac{c(t^4-1) \pm c\sqrt{t^8 - 2t^4 + 1 + 4t^4}}{2t^3}$

$= \frac{c(t^4-1) \pm c\sqrt{(t^4+1)^2}}{2t^3}$

Q. (8)(a)(ii) [continued]:

$$x = \frac{c(t^4-1) \pm c(t^4+1)}{2t^3}$$

$$x = \frac{c[t^4-1+t^4+1]}{2t^3} \quad \text{or} \quad x = \frac{c[t^4-1]-[t^4+1]}{2t^3}$$

$$x = ct \qquad\qquad = -\frac{2c}{2t^3}$$

$$= -\frac{c}{t^3} \qquad \underline{2}$$

→ As $x = ct$ is at P, x co-ordinate of Q $= -\frac{c}{t^3}$.

Note: As we knew $x = ct$ was one point of intersection at start, using quadratic equation (1):

$t^3x^2 - cx(t^4-1) - c^2t = 0$

& $\alpha = ct$ is one root.

Could do $\alpha\beta = \frac{c}{a}$

$$ct \times \beta = \frac{-c^2t}{t^3}$$

$$ct \times \beta = \frac{-c^2}{t^3}$$

$$\therefore \beta = \frac{-c}{t^3}$$

y co-ordinate of Q: $xy = c^2 \Rightarrow y = \frac{c^2}{x}$

$$= \frac{c^2}{1} \times \frac{-t^3}{c}$$

$$y = -ct^3$$

$Q = \left(-\frac{c}{t^3}, -ct^3\right)$.

Ext. 2 Trial paper p. 22

Q.(8)(a)(iii) Co-ordinates of M

[midpoint of PQ]

$$= \left(\frac{ct - \frac{c}{t^3}}{2}, \frac{\frac{c}{t} - ct^3}{2} \right)$$

$$= \left(\frac{ct^4 - c}{2t^3}, \frac{c - ct^4}{2t} \right)$$

$$= \left(\frac{c(t^4-1)}{2t^3}, \frac{-c(t^4-1)}{2t} \right). \quad 1$$

(iv) Cartesian equation for locus of M:

From (iii) $y = -t^2 x \Rightarrow -\frac{y}{x} = t^2$. (1)

$$x^2 = \frac{c^2(t^4-1)^2}{4t^6} \quad (2)$$

$\times$ (2) by $4t^6$

$$4t^6x^2 = c^2(t^8 - 2t^4 + 1) \quad (3)$$

As $t^2 = -\frac{y}{x}$, sub. (1) in (3):

$$4\left(\frac{-y}{x}\right)^3 x^2 = c^2\left(\frac{y^4}{x^4} - \frac{2y^2}{x^2} + 1\right)$$

$$-\frac{4y^3}{x} = c^2\left(\frac{y^4}{x^4} - \frac{2y^2}{x^2} + 1\right)$$

Multiply BS by x^4:

$$-4y^3x^3 = c^2(y^4 - 2x^2y^2 + x^4)$$

$$c^2(x^2 - y^2)^2 + 4x^3y^3 = 0 \quad 2$$

Locus of $M = c^2(x^2 - y^2) + 4x^3y^3 = 0$

Q. (8)(b) $A_1 = \alpha + \beta = \frac{-b}{a}$ $= 1.$

$A_2 = \alpha^2 + \beta^2$
$= (\alpha+\beta)^2 - 2\alpha\beta$
$= 1^2 - 2\times 1$
$= -1.$

$\alpha\beta = \frac{c}{a}$
$= 1.$

To find $A_n = \alpha^n + \beta^n$.

As α, β are roots of $x^2 - x + 1 = 0$

$\alpha^2 - \alpha + 1 = 0$

$\beta^2 - \beta + 1 = 0$

α, β are also roots of $x^{n-2}(x^2 - x + 1) = 0$

[Other root $= 0$] i.e. $x^n - x^{n-1} + x^{n-2} = 0$

So $\alpha^n - \alpha^{n-1} + \alpha^{n-2} = 0$ +

$\beta^n - \beta^{n-1} + \beta^{n-2} = 0$

$(\alpha^n + \beta^n) - (\alpha^{n-1} + \beta^{n-1}) + (\alpha^{n-2} + \beta^{n-2}) = 0$

i.e. $A_n - A_{n-1} + A_{n-2} = 0$

$A_n = A_{n-1} - A_{n-2}$.

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(ii) $A_1 = 1 = 2\cos\frac{\pi}{3}$. $\rightarrow$ True for $n=1$.

Step 2: Assume true for $n=k$:

$A_k = 2\cos\frac{k\pi}{3}$.

Step 3: Trying to prove true for $n=k+1$

i.e. $A_{k+1} = 2\cos\left[\frac{(k+1)\pi}{3}\right]$.

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We know $A_{k+1} = A_k - A_{k-1}$ [from Part (i)].

$= 2\cos\frac{k\pi}{3} - 2\cos\frac{(k-1)\pi}{3}$

$= 2\cos\frac{k\pi}{3} - 2\left[\cos\left(\frac{k\pi}{3} - \frac{\pi}{3}\right)\right]$

$= 2\cos\frac{k\pi}{3} - 2\left[\cos\frac{k\pi}{3}\cos\frac{\pi}{3} + \sin\frac{k\pi}{3}\sin\frac{\pi}{3}\right]$

$= 2\cos\frac{k\pi}{3} - \cos\frac{k\pi}{3} - 2\sin\frac{k\pi}{3}\sin\frac{\pi}{3}$

As $\cos\frac{\pi}{3} = \frac{1}{2}$ $\quad = \cos\frac{k\pi}{3} - 2\sin\frac{k\pi}{3}\sin\frac{\pi}{3}$ (1)

Q. (8)(b)(ii) [continued]: Checking to see if this is equal to $2\cos\left[\frac{(k+1)\pi}{3}\right]$

$$2\cos\left[\frac{(k+1)\pi}{3}\right]$$

$$= 2\left[\cos\left(\frac{k\pi}{3} + \frac{\pi}{3}\right)\right]$$

$$= 2\left(\cos\frac{k\pi}{3}\cos\frac{\pi}{3} - \sin\frac{k\pi}{3}\sin\frac{\pi}{3}\right)$$

$$= \cos\frac{k\pi}{3} - 2\sin\frac{k\pi}{3}\sin\frac{\pi}{3}$$

$$= (1) = LHS.$$

$\therefore$ True for all positive integers k.